After expanding the inequality reduces to
$2(x^3+y^3+z^3)+x^2+y^2+z^2+3(xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+xy+yz+zx+6xyz.$

Since $x^2+y^2+z^2 \geq xy+yz+zx$, it remains to prove that
$2(x^3+y^3+z^3)+3(xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+6xyz.$

This follows again by using the AM-GM inequality properly:
$2(x^3+y^3+z^3)+3(xy^2+yz^2+zx^2) = 2(x^3+xy^2)+2(y^3+yz^2)+2(z^3+zx^2)+(xy^2+yz^2+zx^2) \geq 4x^2y+4y^2z+4z^2x+(xy^2+yz^2+zx^2) =$
$3(x^2y+y^2z+z^2x)+(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2) \geq 3(x^2y+y^2z+z^2x)+6xyz.$

Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Kevin Soto Palacios, Huarmey, Perú; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5486: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $(x_n)_{n\geq 0}$ be the sequence defined by $x_0=0, x_1=1, x_2=1$ and
$x_{n+3}=x_{n+2}+x_{n+1}+x_n+n,\ \forall n\geq 0$. Prove that the series $\sum_{n=1}^{\infty}\frac{x_n}{2^n}$ converges and find its sum.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

The recurrence sequence may be unmasked by generating functions. Let $F(z)$ be the associated generating function. That is, $F(z)=\sum_{n=0}^{\infty}x_nz^n$. Multiplying by z^{n+3} the recurrence relation defining (x_n) and taking into account the initial values it is obtained that

$$F(z)-\left(z+z^2\right)=z\left(F(z)-z\right)+z^2F(z)+z^3F(z)+\frac{z^4}{(1-z)^2}$$

from where $F(z)=\dfrac{z(1-z)^2+x^4}{(z-1)^2(1-z-z^2-z^3)}$.

Since $F(z)$ converges for $|z|<\dfrac{1}{3}\left(\sqrt[3]{17+3\sqrt{33}}-\dfrac{2}{\sqrt[3]{17+3\sqrt{33}}}-1\right)\sim 0.5436...$, then $\sum_{n=1}^{\infty}\frac{x_n}{2^n}=F(1/2)=6.$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Answer: 6.

Clearly x_n increases and $x_n \geq 1$.

$$\sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{x_1}{2} + \frac{x_2}{4} + \sum_{n=3}^{p} \frac{x_n}{2^n} = \frac{3}{4} + \sum_{n=0}^{p-3} \frac{x_{n+3}}{2^{n+3}} =$$

$$= \frac{3}{4} + \sum_{n=0}^{p-3} \left(\underbrace{\frac{x_{n+2}}{2^{n+3}}}_{I_1} + \underbrace{\frac{x_{n+1}}{2^{n+3}}}_{I_2} + \underbrace{\frac{x_n}{2^{n+3}}}_{I_3} \right) + \sum_{n=0}^{p-3} \frac{n}{2^{n+3}} =$$

$$= \frac{3}{4} + \underbrace{\sum_{n=2}^{p-1} \frac{x_n}{2^{n+1}}}_{I_1} + \underbrace{\sum_{n=1}^{p-2} \frac{x_n}{2^{n+2}}}_{I_2} + \underbrace{\sum_{n=1}^{p-3} \frac{x_n}{2^{n+3}}}_{I_3} + \sum_{n=0}^{p-3} \frac{n}{2^{n+3}} =$$

$$= \frac{3}{4} + \underbrace{-\frac{1}{4} + \sum_{n=1}^{p} \frac{x_n}{2^{n+1}} - \frac{x_p}{2^{p+1}}}_{I_1} + \underbrace{\sum_{n=1}^{p} \frac{x_n}{2^{n+2}} - \frac{x_{p-1}}{2^{p+1}} - \frac{x_p}{2^{p+2}}}_{I_2} +$$

$$+ \underbrace{\sum_{n=1}^{p} \frac{x_n}{2^{n+3}} - \frac{x_{p-2}}{2^{p+1}} - \frac{x_{p-1}}{2^{p+2}} - \frac{x_p}{2^{p+3}}}_{I_3} + \underbrace{\sum_{n=0}^{p-3} \frac{n}{2^{n+3}}}_{\to 1/4 \text{ as } p \to \infty}$$

It follows

$$\frac{1}{8} \sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{1}{2} - \left[\frac{x_p}{2^{p+1}} + \frac{x_{p-1}}{2^{p+1}} + \frac{x_p}{2^{p+2}} + \frac{x_{p-2}}{2^{p+1}} + \frac{x_{p-1}}{2^{p+2}} \right] + \frac{1}{4}$$

Now we prove the

Lemma $x_k/2^k \to 0$.

Proof of the Lemma

First step: the sequence $x_k/2^k$ in monotonic not increasing.

$$\frac{x_{k+3}}{2^{k+3}} = \frac{x_{k+2} + x_{k+1} + x_k + k}{2^{k+3}} \leq \frac{x_{k+2}}{2^{k+2}} \iff \frac{x_{k+1} + x_k + k}{2^{k+3}} \leq \frac{x_{k+2}}{2^{k+3}}$$

that is

$$x_{(k-1)+2} + x_{(k-1)+1} + (k-1) + 1 \leq x_{(k-1)+3}$$

and this is implied by

$$x_{(k-1)+2} + x_{(k-1)+1} + (k-1) + 1 \leq x_{(k-1)+2} + x_{(k-1)+1} + x_{k-1} + (k-1) = x_{(k-1)+3}$$

via $x_{k-1} \geq 1$. The monotonicity of the sequence means that the limit L of $x_k/2^k$ does exist and moreover $0 \leq L < +\infty$. If $L = 0$ the proof is concluded yielding

$$\lim_{p \to \infty} \frac{1}{8} \sum_{n=1}^{p} \frac{x_n}{2^n} = \frac{3}{4} \iff \sum_{n=1}^{\infty} \frac{x_n}{2^n} = 6$$

$L \neq 0$ is impossible as shown by the following argument. We employ the Cesaro–Stolz theorem that states:

$$\lim_{k \to \infty} \frac{x_k}{2^k} = \lim_{k \to \infty} \frac{x_{k+1} - x_k}{2^{k+1} - 2^k}$$

provided that the second limit does exist. We write

$$\frac{x_{k+3}-x_{k+2}}{2^{k+3}-2^{k+2}}=\frac{x_{k+1}+x_k+k}{2^{k+2}}=\frac{1}{2}\frac{x_{k+1}}{2^{k+1}}+\frac{1}{4}\frac{x_k}{2^k}+\frac{k}{2^{k+2}}$$

The existence of the limit $L=\lim\limits_{k\to\infty}\frac{x_k}{2^k}$ would imply

$$L=\frac{1}{2}L+\frac{1}{4}L \implies L=0$$

Solution 3 by Arkady Alt, San Jose, CA

For any sequence $(x_n)_{n\geq 0}$ let $T(x_n) := x_{n+3}-x_{n+2}-x_{n+1}-x_n, n\in N\cup\{0\}$.
Obvious that such defined operator T (we will call it Tribonacci Operator) is linear.
Since $T\left(-\frac{n}{2}\right)=-\frac{n+3}{2}+\frac{n+2}{2}+\frac{n+1}{2}+\frac{n}{2}=n$ then denoting
$u_n := x_n+\frac{n}{2}, n\in N\cup\{0\}$
we obtain $x_n=u_n-\frac{n}{2}, n\in N\cup\{0\}$ where $T(u_n)=0$ and,
$u_0=0, u_1=1+\frac{1}{2}=\frac{3}{2}, u_2=1+\frac{2}{2}=2.$
Let $(t_n)_{n\geq 0}$ be the sequence defined by $t_0=0, t_1=1, t_2=1$ and
$T(t_n)=0, n\in N\cup\{0\}$.
(Tribonacci Sequence). We have $t_3=2, t_4=4, t_5=7, t_6=13, t_7=24, t_8=44, \ldots$
Since $\det\begin{pmatrix}0&1&1\\1&1&2\\1&2&4\end{pmatrix}\neq 0$ then for any sequence $(x_n)_{n\geq 0}$ there is triple (c_1,c_2,c_3) of real numbers such that $x_n=c_2t_n+c_2t_{n+1}+c_3t_{n+2}$, that is sequences $(t_n)_{n\geq 0}$, $(t_{n+1})_{n\geq 0}$, $(t_{n+2})_{n\geq 0}$
form a basis of 3-dimesion space $\ker T := \{(x_n)_{n\geq 0} \mid T(x_n)=0, n\in N\cup\{0\}\}$.
We will find representation u_n as linear combination of t_n, t_{n+1}, t_{n+2},
namely, $u_n=c_1t_n+c_2t_{n+1}+c_3t_{n+2}$, $n\in N\cup\{0\}$.
We
have $u_0=c_1t_0+c_2t_1+c_3t_2 \iff c_2+c_3=0, u_1=c_1t_1+c_2t_2+c_3t_3 \iff c_1+c_2+2c_3=\frac{3}{2}$,
$u_2=c_1t_2+c_2t_3+c_3t_4 \iff c_1+2c_2+4c_3=2$. From this system of equations we obtain
$c_3=-c_2, c_1-c_2=\frac{3}{2}, c_1-2c_2=2$.Hence, $c_1=1, c_2=-\frac{1}{2}, c_3=\frac{1}{2}$ and since
$u_n=t_n-\frac{t_{n+1}}{2}+\frac{t_{n+2}}{2}$ we obtain $x_n=t_n-\frac{t_{n+1}}{2}+\frac{t_{n+2}}{2}-\frac{n}{2}=\frac{2t_n-t_{n+1}+t_{n+2}-n}{2}$.
Since radius of convergence of seies $\sum\limits_{n=1}^{\infty} nx^{n-1}$ is 1 and $\sum\limits_{n=1}^{\infty} nx^{n-1}=\frac{1}{(1-x)^2}$
then $\sum\limits_{n=1}^{\infty}\frac{n}{2^n}=\frac{1}{2}\sum\limits_{n=1}^{\infty}\frac{n}{2^{n-1}}=\frac{1}{2}\frac{1}{(1-1/2)^2}=2$ and, therefore, for convergency of $\sum\limits_{n=1}^{\infty}\frac{x_n}{2^n}$ suffice to prove convergency of series $\sum\limits_{n=1}^{\infty}\frac{t_n}{2^n}$.
We can prove that using another basis of $\ker T$ which form sequences $(\alpha^n)_{n\geq 0}, (\beta^n)_{n\geq 0}$, $(\gamma^n)_{n\geq 0}$
where α,β,γ are roots of characteristic equation $x^3-x^2-x-1=0$.
Substitution $x=\frac{4u+1}{3}$ in equation $x^3-x^2-x-1=0$ give us equivalent equation
$4u^3-3u=\frac{19}{8}$

which we solve using substitution $u := \frac{1}{2}\left(t+\frac{1}{t}\right)$. Then equation $4u^3 - 3u = \frac{19}{8}$ becomes $4\left(\frac{1}{2}\left(t+\frac{1}{t}\right)\right)^3 - 3\cdot\frac{1}{2}\left(t+\frac{1}{t}\right) = \frac{19}{8} \iff \frac{1}{t^3}+t^3 = \frac{19}{4}$.Denoting $z := t^3$ we obtain

$\frac{1}{z}+z = \frac{19}{4} \iff z = \frac{19-3\sqrt{33}}{8}, \frac{19+3\sqrt{33}}{8} \iff t^3 = \frac{19-3\sqrt{33}}{8}, \frac{19+3\sqrt{33}}{8}$.

Since $\frac{19-3\sqrt{33}}{8}\cdot\frac{19+3\sqrt{33}}{8} = 1$ and $u = \frac{1}{2}\left(t+\frac{1}{t}\right)$ then suffices to find $t^3 = \frac{19+3\sqrt{33}}{8}$.

We have $t = r(\cos\varphi + i\sin\varphi)$, where $r = \frac{\sqrt[3]{19+3\sqrt{33}}}{2}$ and $\varphi = \frac{2k\pi}{3}, k = 1,2,3$.

that is $t_k = \frac{\sqrt[3]{19+3\sqrt{33}}}{2}\omega^k, k = 1,2,3$ and $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \omega^3 = 1$.

Thus, denoting $\theta := \sqrt[3]{19+3\sqrt{33}}, \theta^* := \sqrt[3]{19-3\sqrt{33}}$ we obtain

$\alpha = \frac{1+\theta+\theta^*}{3}, \beta = \frac{1+\omega\theta+\omega^2\theta^*}{3}$,

$\gamma = \frac{1+\omega^2\theta+\omega\theta^*}{3}$, the three roots of the equation $x^3 - x^2 - x - 1 = 0$.

We will prove that $\alpha = \frac{1+\theta+\theta^*}{3} < 2$.

First note that by Power Mean–Arithmetic Mean inequality

$p := \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} < 2\sqrt[3]{\frac{19+3\sqrt{33}+19-3\sqrt{33}}{2}} = 2\sqrt[3]{19} < 2\sqrt[3]{27} = 6$.

Since $\sqrt[3]{19+3\sqrt{33}}\cdot\sqrt[3]{19-3\sqrt{33}} = \sqrt[3]{19^2 - 9\cdot 33} = 4$ then

$p^3 = 38 + 3\sqrt[3]{19+3\sqrt{33}}\cdot\sqrt[3]{19-3\sqrt{33}}\cdot p = 38 + 12p < 38 + 12\cdot 6 = 110 < 125 = 5^3$.

Hence, $\alpha < 2$. Also, we obtain $|\beta|, |\gamma| \le \frac{1+\theta+\theta^*}{3} < 2$.

Since series $\sum_{n=1}^{\infty}\left(\frac{\alpha}{2}\right)^n, \sum_{n=1}^{\infty}\left(\frac{\beta}{2}\right)^n, \sum_{n=1}^{\infty}\left(\frac{\gamma}{2}\right)^n$ are convergent and t_n is linear combination of

$(\alpha^n)_{n\ge 0}, (\beta^n)_{n\ge 0}, (\gamma^n)_{n\ge 0}$ then series $\sum_{n=1}^{\infty}\frac{t_n}{2^n}$ convergent as well.

Now we ready to find sum of series $\sum_{n=1}^{\infty}\frac{x_n}{2^n}$.

Let $s_n := \sum_{k=1}^{n}\frac{t_n}{2^n}$ and $s(x) = \sum_{k=0}^{n} t_{n+1}x^n$. Note also that function $\frac{1}{1-x-x^2-x^3}$ generates

Tribonacci numbers. Indeed, let $\frac{1}{1-x-x^2-x^3} = \sum_{n=0}^{\infty} a_n x^n$. Then

$\sum_{n=0}^{\infty} a_n x^n \cdot \left(1-x-x^2-x^3\right) = 1$

and since

$\sum_{n=0}^{\infty} a_n x^n \cdot \left(1-x-x^2-x^3\right) = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} -_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+3} =$

$a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + \sum_{n=3}^{\infty}(a_{n+3} - a_{n+2} - a_{n+1} - a_n)x^{n+3}$ then

$a_0 = 1, a_1 - a_0 = a_2 - a_1 - a_0 = 0$ implies $a_1 = 1, a_2 = 2$ and

$a_{n+3} - a_{n+2} - a_{n+1} - a_n = 0, n \in N \cup \{0\}$. Thus, $a_n = t_{n+1}, n \in N \cup \{0\}$ and, therefore, $\sum\limits_{k=0}^{n} t_{n+1}x^n = s(x) = \dfrac{1}{1 - x - x^2 - x^3}$. In,
particular, $\sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} = \dfrac{1}{2}\sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^{n-1-}} = \dfrac{1}{2}s\left(\dfrac{1}{2}\right) =$
$\dfrac{1}{2} \cdot \dfrac{1}{1 - \dfrac{1}{2} - \left(\dfrac{1}{2}\right)^2 - \left(\dfrac{1}{2}\right)^3} = 4.$
Then, $\sum\limits_{n=1}^{\infty} \dfrac{x_n}{2^n} = \sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} - \sum\limits_{n=1}^{\infty} \dfrac{t_{n+1}}{2^{n+1}} + 2\sum\limits_{n=1}^{\infty} \dfrac{t_{n+2}}{2^{n+2}} - \sum\limits_{n=1}^{\infty} \dfrac{n}{2^{n+1}} =$
$\sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} - \sum\limits_{n=2}^{\infty} \dfrac{t_n}{2^n} + 2_{n=3}^{\infty}\dfrac{t_n}{2^n} - \dfrac{1}{2}\sum\limits_{n=1}^{\infty} \dfrac{n}{2^n} =$
$\dfrac{t_1}{2^1} + 2\sum\limits_{n=3}^{\infty} \dfrac{t_n}{2^n} - \dfrac{1}{2} \cdot 2 = -\dfrac{1}{2} + 2\sum\limits_{n=1}^{\infty} \dfrac{t_n}{2^n} - 2\left(\dfrac{t_1}{2^1} + \dfrac{t_2}{2^2}\right) = -\dfrac{1}{2} + 2 \cdot 4 - 2\left(\dfrac{1}{2} + \dfrac{1}{4}\right) = 6.$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the given series converges by first using induction to prove that $x_n < 1.95^n$ for each positive integer n. Note that this claim holds for $n \in \{1, 2, 3\}$. Given a positive integer k, if $x_n < 1.95^n$ for $n \in \{k, k+1, k+2\}$, then

$$x_{k+3} < 1.95^{k+2} + 1.95^{k+1} + 1.95^k + k = 1.95^k(6.7525) + k.$$

Thus it suffices to show that $1.95^k(6.7525) + k \le 1.95^{k+3}$, or equivalently $k \le 1.95^k(0.662375)$. This latter inequality holds for each positive integer k (using a separate induction argument). Hence $x_n < 1.95^n$ for $n \ge 1$, so for any positive integer m,

$$\sum_{n=1}^{m} \frac{x_n}{2^n} < \sum_{n=1}^{\infty} \frac{x_n}{2^n} < \sum_{n=1}^{\infty} \frac{1.95^n}{2^n} = \frac{0.975}{1 - 0.975} = 39.$$

Since its sequence of partial sums is increasing and bounded above, the given series converges.

Next, we let $\sum\limits_{n=1}^{\infty} \dfrac{x_n}{2^n} = L$. Then

$$L = \frac{1}{2} + \frac{1}{4} + \sum_{n=0}^{\infty} \frac{x_{n+2} + x_{n+1} + x_n + n}{2^{n+3}} = \frac{3}{4} + \frac{1}{2}\left(L - \frac{1}{2}\right) + \frac{1}{4}L + \frac{1}{8}L + \sum_{n=0}^{\infty} \frac{n}{2^{n+3}}.$$

Since $\sum\limits_{n=0}^{\infty} \dfrac{n}{2^n} = 2$, we conclude $L = \frac{7}{8}L + \frac{1}{2} + \frac{1}{8}(2)$ and hence $L = 6$.

Also solved by Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler (two solutions), Herrliberg, Switzerland; David Stone and John Hawkins, Southern Georgia University, Statesboro, GA, and the proposer.

Mea – Culpa